Homework 5 2024-2025

The Chinese University of Hong Kong
Department of Mathematics
MMAT 5340 Probability and Stochastic Analysis
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Please submit your solutions on blackboard before
11:59 AM, Feb 24th 2025

February 17, 2025

1.

Let $(\xi_k)_{k\geq 1}$ be a sequence of independent and identically distributed random variables with standard Gaussian distribution, i.e. $\xi_k \sim N(0,1)$. We define $X = (X_n)_{n\geq 0}$ as follows:

$$X_0 := 0, \quad X_n := \sum_{k=1}^{n} \frac{1}{k}\xi_k, \quad \text{for all } n \geq 1.$$

(a) Prove that X is a martingale.

(b) Prove that

$$\sup_{n\in\mathbb{N}} E[|X_n|^2] < \infty.$$

(c) By the convergence theorem of the martingale (Theorem 2.4), we know that $X_n \to X_\infty$ a.s. and in L^2 for some random variable X_∞ as $n \to \infty$.

i. Compute the characteristic function ψ_n of X_n, where ψ_n is defined as

$$\psi_n(\theta) := E[e^{i\theta X_n}], \quad \theta \in \mathbb{R}.$$

ii. Compute

$$\psi(\theta) := \lim_{n\to\infty} \psi_n(\theta), \quad \theta \in \mathbb{R}.$$

iii. Identify the distribution of X_∞.
Hint: ψ is the characteristic function of X_∞ and the distribution of a random variable is uniquely determined by its characteristic function.

2.

Let $(\xi_k)_{k\geq 1}$ be a sequence of independent and identically distributed random variables such that $P[\xi_k = \pm 1] = \frac{1}{2}$. We define $X = (X_n)_{n\geq 0}$ as follows:

$$X_0 := 0, \quad X_n := \sum_{k=1}^{n} 2^{k-1}\xi_k 1_{\{k\leq\tau\}}, \quad \text{where } \tau := \inf\{k \in \mathbb{N} : \xi_k = 1\}.$$

(a) Prove that X is a martingale.

(b) Compute $P[\tau > n]$ and deduce that $P[\tau < +\infty] = 1$.

Hint: $\{\tau > n\} = \{\xi_1 = \cdots = \xi_n = -1\}$.

(c) Prove that $X_\tau = 1$ a.s.

Remark: It may be worth noting that in this case, we have $1 = E[X_\tau] \neq E[X_0] = 0$.

(d) Compute $E[|X_n|]$ and prove that $\sup_{n\in\mathbb{N}} E[|X_n|] < \infty$, and $\lim_{n\to\infty} X_n = X_\tau$ a.s.

Hint:

$$E[|X_n|] = E[|X_n|1_{\{\tau>n\}}] + E[|X_\tau|1_{\{\tau\leq n\}}].$$